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SECOND ORDER PARABOLIC EQUATIONS IN BANACH SPACES WITH DYNAMIC BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we exhibit a unified treatment of the mixed initial boundary value problem for second order (in time) parabolic linear differential equations in Banach spaces, whose boundary conditions are of a dynamical nature. Results regarding existence, uniqueness, continuous dependence (on initial data) and regularity of classical and strict solutions are established. Moreover, several examples are given as samples for possible applications.

1. INTRODUCTION

Of concern is the inhomogeneous complete second order differential equation

$$u''(t) + Au(t) + Bu'(t) = f(t), \quad t > 0, \tag{1.1}$$

in a Banach space E, where A and B are linear operators in E, and f an E-valued function. The Cauchy problem for (1.1) has been extensively studied since the end of 1950s (see H. O. Fattorini [9, 10] and T. J. Xiao and J. Liang [24, 25] for surveys).

In this paper, we consider a mixed initial boundary value problem for (1.1), in which besides the usual initial condition

$$u(0) = u_0, \quad u'(0) = u_1, \tag{1.2}$$

there is also a boundary condition given by

$$x''(t) + A_1x(t) + B_1x'(t) = G_0u(t) + G_1u'(t) + g(t), \quad t > 0. \tag{1.3}$$

Here $x(\cdot)$ stands for the boundary value of the state function $u(\cdot)$, these two functions being connected by a linear boundary operator P (from $\mathcal{D}(A)$ to another Banach space X),

$$x(t) = Pu(t), \quad t > 0; \tag{1.4}$$

A_1 and B_1 are linear operators in X, g an X-valued function, and G_i $(i = 0, 1)$ are linear operators (feedback operators) from $\mathcal{D}(G_i) \subset E$ to X. The boundary condition (1.3) is of a dynamic nature, for which we initially have

$$x(0) = x_0, \quad x'(0) = x_1. \tag{1.5}$$

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Dynamic boundary conditions occur in diverse practical problems, for instance, in those modelling the dynamic vibrations of linear viscoelastic rods and beams with tip masses attached at their free ends; see, e.g., [2, 4, 21]. The study of evolution equations with dynamic boundary conditions from the mathematical point of view dates back to 1961, when J. L. Lions [18, p. 117, 118] treated such equations and gave weak solutions by means of the variational method. Since then, this issue has been investigated to a large extent (see, e.g., [5, 6, 8, 11, 12, 13, 15, 16, 17, 18, 22] and references therein). While most of the previous research concerns the case of first order in time, there has been little regarding the second order (in time) case. It seems nontrivial or impractical, as far as dynamic boundary conditions are concerned, to get solutions for a second order (in time) problem by reducing it to a first order one, especially to obtain strong solutions with high time regularity and spatial regularity (cf. [1, Theorem 2.1 and Remark 4.2]). In the present paper, we shall deal with the second order problem (1.1) - (1.5) in a direct way, without reduction. This approach will yield strong solutions with desirable regularity, as well as build up theorems of a general nature.

To begin, write

$$\mathbb{A} := \begin{pmatrix} A & 0 \\ -G_0 & A_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(G_0)) \times \mathcal{D}(A_1); \quad x = Pu \right\},$$

$$\mathbb{B} := \begin{pmatrix} B & 0 \\ -G_1 & B_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{B}) := (\mathcal{D}(B) \cap \mathcal{D}(G_1)) \times \mathcal{D}(B_1),$$

$$y(t) := \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad h(t) := \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad y_0 := \begin{pmatrix} u_0 \\ x_0 \end{pmatrix}, \quad y_1 := \begin{pmatrix} u_1 \\ x_1 \end{pmatrix}.$$

Then, problem (1.1) - (1.5) is converted into an abstract Cauchy problem in the product space $\mathbf{E} := E \times X$:

$$\begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = h(t), \quad t > 0, \\ y(0) = y_0, \ y'(0) = y_1. \end{cases}$$

How can one deal with this problem involving two operator matrices? We shall present some ideas about it. This paper is confined to equations of parabolic type, and those of hyperbolic type will be considered in a forthcoming paper.

In order to carry out our strategy, we still need to introduce another boundary operator P_1, a linear operator from $\mathcal{D}(B)$ to the quotient space X/X_0 (X_0 is a closed linear subspace of X). The P_1 can be chosen flexibly in applications (see Examples 4.1, 4.3 and 4.5), such that the relation

$$x'(t) \in P_1 u'(t), \quad t > 0, \tag{1.6}$$

is implied by (1.1), (1.3) and (1.4). The simplest P_1 is in the case of $X_0 = X$.

For the two operators A and B in the state space E, we define

$$A_0 := A\Big|_{\ker P}, \quad B_0 := B\Big|_{\ker P_1}. \tag{1.7}$$

Then the elements in the domains of A_0 and B_0 have zero boundary values in some sense. A condition of parabolic type will be given on the operator pair (A_0, B_0) (also on (A_1, B_1)), which is easy to verify in concrete situations. Moreover, for equations (1.1) and (1.3), we regard A, B, A_1, and B_1 as principal operators to which G_0 and G_1 are subordinated. For a wider applicability, we take four more perturbing (linear) operators into consideration:

$$\widetilde{A}:\mathcal{D}(\widetilde{A})\subset E\to E,\quad \widetilde{B}:\mathcal{D}(\widetilde{B})\subset E\to E,$$
$$\widetilde{A}_1:\mathcal{D}(\widetilde{A}_1)\subset X\to X,\quad \widetilde{B}_1:\mathcal{D}(\widetilde{B}_1)\subset X\to X.$$

Thus, we shall actually study

$$\begin{cases} y''(t)+\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y(t)+\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y'(t)=h(t), & t>0,\\ y(0)=y_0,\ y'(0)=y_1 \end{cases} \tag{1.8}$$

in space $\mathbf{E}$, with the main operator matrices $\mathbf{A}$, $\mathbf{B}$ and the perturbing operators $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ defined as follows:

$$\mathbf{A}:=\begin{pmatrix} A & 0\\ 0 & A_1\end{pmatrix},\quad \mathcal{D}(\mathbf{A}):=\left\{\begin{pmatrix} u\\ x\end{pmatrix}\in\mathcal{D}(A)\times\mathcal{D}(A_1);\quad x=Pu\right\},$$

$$\mathbf{B}:=\begin{pmatrix} B & 0\\ 0 & B_1\end{pmatrix},\quad \mathcal{D}(\mathbf{B}):=\left\{\begin{pmatrix} u\\ x\end{pmatrix}\in\mathcal{D}(B)\times\mathcal{D}(B_1);\quad x\in P_1u\right\}.$$

$$\widetilde{\mathbf{A}}:=\begin{pmatrix} \widetilde{A} & 0\\ -G_0 & \widetilde{A}_1\end{pmatrix},\quad \widetilde{\mathbf{B}}:=\begin{pmatrix} \widetilde{B} & 0\\ -G_1 & \widetilde{B}_1\end{pmatrix}.$$

In Section 2, we shall show under suitable conditions that the operator pair $\left(\mathbf{A}+\widetilde{\mathbf{A}},\,\mathbf{B}+\widetilde{\mathbf{B}}\right)$ possesses certain parabolicity (Theorem 2.3), and then construct an operator function $\widetilde{\mathbf{S}}(\cdot)$ (a fundamental solution operator of (1.8)) having a holomorphic extension to a sector Σ_θ ($\theta\in(0,\frac{\pi}{2}]$) and satisfying various nice properties (Theorem 2.4). Making use of this, we will formulate and prove, in Section 3, our main theorem (Theorem 3.3) with regard to the existence and uniqueness of classical and strict solutions for (1.8), also continuous dependence (on initial data) and regularity of the solutions. Finally, in Section 4 we shall exhibit three applications of our theorems to damped beam and plate-like equations with dynamic boundary conditions.

Notation. For Banach spaces E and X, $\mathcal{L}(E,\ X)$ is the space of all bounded linear operators from E into X. The space $\mathcal{L}(E,\ E)$ is abbreviated to $\mathcal{L}(E)$. For a linear operator A in E, $\mathcal{D}(A)$, $\ker A$, and $\rho(A)$ stands for its domain, kernel, and resolvent set, respectively. The operator $A\big|_Z$ means the restriction of A to a space Z. $C^\alpha([0,T];E)$ ($\alpha\in(0,1)$) is the Banach space of Hölder continuous functions $q:[0,T]\to E$ with exponent α and norm given by

$$\sup_{0\le t\le T}\|q(t)\|_E+\sup_{0\le s<t\le T}(t-s)^{-\alpha}\|q(t)-q(s)\|_E.$$

Write

$$\Sigma_\theta:=\{\lambda\in\mathbf{C};\quad \lambda\neq0,\ |\arg\lambda|<\theta\},\quad \theta\in(0,\pi],$$
$$R_i(\lambda):=(\lambda^2+A_i+\lambda B_i)^{-1},\quad i=0,1,\quad \lambda\in\mathbf{C},$$
$$\widetilde{\mathbf{R}}(\lambda):=\left(\lambda^2+\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)+\lambda\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)\right)^{-1},\quad \lambda\in\mathbf{C},$$

if the inverse operators exist, and

$$\rho(A_0,B_0):=\{\lambda\in\mathbf{C};\ R_0(\lambda)\text{ exists and belongs to }\mathcal{L}(E)\}.$$

By $[\mathcal{D}(A)]$ we denote the space $\mathcal{D}(A)$ equipped with the graph norm, $[\mathcal{D}(A)]_P$ the space $\mathcal{D}(A)$ with the norm

$$\|u\|_{A,P} := \|u\| + \|Au\| + \|Pu\|,$$

$[\mathcal{D}(B)]_{P_1}$ the space $\mathcal{D}(B)$ with the norm

$$\|u\|_{B,P_1} := \|u\| + \|Bu\| + \|P_1 u\|_{X/X_0},$$

$[\mathcal{D}(A) \cap \mathcal{D}(B)]$ the space $\mathcal{D}(A) \cap \mathcal{D}(B)$ with the norm

$$\|u\|_{A,B} := \|u\| + \|Au\| + \|Bu\|,$$

and $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$ the space $\mathcal{D}(A) \cap \mathcal{D}(B)$ with the norm

$$\|u\|_{A,B,P} := \|u\| + \|Au\| + \|Bu\| + \|Pu\|.$$

2. Parabolicity

We first give some basic properties of the operators A, B and P.

Lemma 2.1. *Suppose that the following* (H_1) *is satisfied.*

(H_1) $[\mathcal{D}(A)]_P$ *and* $[\mathcal{D}(B)]_{P_1}$ *are complete,* $P(\mathcal{D}(A) \cap \mathcal{D}(B)) = X$*, and* $Pu \in P_1 u$ *for any* $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$.

Then

(1) *The space* $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$ *is complete.*

(2) *If* $\lambda \in \rho(A_0, B_0)$*,* $\lambda \neq 0$*, then we have that* $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ *is a bijection of* $\ker(\lambda^2 + A + \lambda B)$ *onto* X*, and*

$$D_\lambda := \left(P\Big|_{\ker(\lambda^2+A+\lambda B)} \right)^{-1}$$

is bounded from X *to* $(\ker(\lambda^2 + A + \lambda B), \|\cdot\|_{A,B,P})$.

(3) *For every* $\lambda, \mu \in \rho(A_0, B_0)$ *with* λ, $\mu \neq 0$,

$$D_\lambda = D_\mu + (\mu - \lambda) R_0(\lambda)(\mu + \lambda + B) D_\mu. \tag{2.1}$$

Proof. (1) Suppose that $\{u_n\}_{n\in N}$ is a Cauchy sequence in $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$. Then it is easy to see that $\{u_n\}_{n\in N}$ is a Cauchy sequence in $[\mathcal{D}(A)]_P$. So there exists $u \in \mathcal{D}(A)$ such that

$$u_n \to u, \quad Au_n \to Au, \quad Pu_n \to Pu, \text{ as } n \to \infty. \tag{2.2}$$

Moreover, $\{u_n\}_{n\in N}$ is also a Cauchy sequence in $[\mathcal{D}(B)]_{P_1}$, because of

$$\|P_1 u_n\|_{X/X_0} \le \|Pu_n\|$$

by (H_1); therefore there is $v \in \mathcal{D}(B)$ such that

$$\lim_{n\to\infty} u_n = v, \quad \lim_{n\to\infty} Bu_n = Bv. \tag{2.3}$$

Combining (2.2) and (2.3) shows that $u = v$, and so

$$u \in \mathcal{D}(B), \quad \lim_{n\to\infty} Bu_n = Bu.$$

This verifies the completeness of $[\mathcal{D}(A) \cap \mathcal{D}(B)]_P$.

(2) Assume that $u, v \in \ker(\lambda^2 + A + \lambda B)$, with $Pu = Pv$. Then

$$(\lambda^2 + A + \lambda B)(u - v) = 0 \quad \text{and} \quad P(u - v) = 0,$$

which implies $P_1(u-v)=0$. Therefore $u-v\in\mathcal{D}(A_0)\cap\mathcal{D}(B_0)$ by the definitions of A_0 and B_0. Thus we have

$$(\lambda^2+A_0+\lambda B_0)(u-v)=0.$$

This yields that $u-v=0$ since $\lambda\in\rho(A_0,B_0)$. Hence $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is injective. Next take $x\in X$. Then there is $u\in\mathcal{D}(A)\cap\mathcal{D}(B)$ such that $Pu=x$, by (H_1). Put

$$v_1=R_0(\lambda)(\lambda^2+A+\lambda B)u,\quad v_2=u-v_1.$$

We see easily that $v_1\in\mathcal{D}(A_0)$ and $(\lambda^2+A+\lambda B)v_2=0$. So

$$Pv_1=0,\quad Pv_2=Pu-Pv_1=x,$$

and $v_2\in\ker(\lambda^2+A+\lambda B)$. This indicates that $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is surjective. Finally, we observe that $(\ker(\lambda^2+A+\lambda B),\|u\|_{A,B,P})$ is a Banach space in view of (1), and $P\Big|_{\ker(\lambda^2+A+\lambda B)}$ is a bounded linear operator from $(\ker(\lambda^2+A+\lambda B),\|\cdot\|_{A,B,P})$ onto X. So an appeal to the open mapping theorem gives the boundedness of D_λ.

(3) Write

$$Q=[I+(\mu-\lambda)R_0(\lambda)(\mu+\lambda+B)]D_\mu.$$

Then for each $x\in X$,

$$\begin{aligned}(\lambda^2+A+\lambda B)Qx &= [(\lambda^2+A+\lambda B)+\mu^2-\lambda^2+(\mu-\lambda)B]D_\mu x\\ &= (\mu^2+A+\mu B)D_\mu x=0,\end{aligned}$$

since $D_\mu x\in\ker(\mu^2+A+\mu B)$. Thus we see $Range(Q)\subset\ker(\lambda^2+A+\lambda B)$. Moreover, we have $PQ=PD_\mu=I$, noting $PR_0(\lambda)=0$. Therefore, we deduce $Q=D_\lambda$ as claimed. The proof is then complete.

The following is the hypotheses of parabolic type on A_0, B_0 (see (1.7)) and on A_1, B_1.

(H_2) The operators A_0 and B_0 are closed, and for each $\varphi\in(0,\theta)$ ($\theta\in(0,\frac{\pi}{2}]$), there exist M_φ, $\omega_\varphi>0$ such that

$$\|\lambda R_0(\lambda)\|,\ \left\|\lambda^{-1}A_0R_0(\lambda)\right\|\le M_\varphi|\lambda|^{-1},\quad \lambda\in\omega_\varphi+\Sigma_{\frac{\pi}{2}+\varphi}.$$

(H_3) The operators A_1 and B_1 are closed, and for each $\varphi\in(0,\theta)$ ($\theta\in(0,\frac{\pi}{2}]$), there exist M_φ, $\omega_\varphi>0$ such that

$$\|\lambda R_1(\lambda)\|,\ \left\|\lambda^{-1}A_1R_1(\lambda)\right\|\le M_\varphi|\lambda|^{-1},\quad \lambda\in\omega_\varphi+\Sigma_{\frac{\pi}{2}+\varphi}.$$

Remark 2.2. In concrete problems, it happens quite often that A_1 and B_1 are bounded operators on X. In this situation, (H_3) holds automatically.

Prior to stating Theorem 2.3 below concerning (among others) the parabolicity of $\left(\mathbf{A}+\widetilde{\mathbf{A}},\mathbf{B}+\widetilde{\mathbf{B}}\right)$, we recall (cf., e.g., [7, p. 169]):

A linear operator $\mathbb{B}$ in a Banach space Y is called $\mathbb{A}$-bounded, for a linear operator $\mathbb{A}$ in Y, if $\mathcal{D}(\mathbb{A})\subset\mathcal{D}(\mathbb{B})$ and there exist constants a, $b>0$ such that

$$\|\mathbb{B}y\|\le a\|\mathbb{A}y\|+b\|y\| \tag{2.4}$$

for all $y\in\mathcal{D}(\mathbb{A})$; the $\mathbb{A}$-bound of $\mathbb{B}$ is

$$\inf\{a>0;\ \text{there is } b>0 \text{ such that (2.4) holds}\}.$$

Theorem 2.3. *Let $\theta \in (0, \frac{\pi}{2}]$. Suppose that* (H_1) – (H_3) *hold. Let*

$$\widetilde{A} \in \mathcal{L}\left([\mathcal{D}(A)]_P, E\right), \quad \widetilde{B} \in \mathcal{L}\left([\mathcal{D}(B)]_{P_1}, E\right), \tag{2.5}$$

$$G_0 \in \mathcal{L}\left([\mathcal{D}(A)]_P, X\right), \quad G_1 \in \mathcal{L}\left([\mathcal{D}(B)]_{P_1}, X\right), \tag{2.6}$$

$$\widetilde{A}_1 \in \mathcal{L}\left([\mathcal{D}(A_1)], X\right), \quad \widetilde{B}_1 \in \mathcal{L}\left([\mathcal{D}(B_1)], X\right), \tag{2.7}$$

such that $\widetilde{A}$, G_0 are A_0-bounded with A_0-bound zero, $\widetilde{B}$, G_1 are B_0-bounded with B_0-bound zero, $\widetilde{A}_1$ is A_1-bounded with A_1-bound zero, and $\widetilde{B}_1$ is B_1-bounded with B_1-bound zero. Then

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *are closed, and*

$$\widetilde{\mathbf{A}} \in \mathcal{L}([\mathcal{D}(\mathbf{A})], \mathbf{E}), \quad \widetilde{\mathbf{B}} \in \mathcal{L}([\mathcal{D}(\mathbf{B})], \mathbf{E}). \tag{2.8}$$

(2) *There exist $M'_\varphi > M_\varphi$, $\omega'_\varphi > \omega_\varphi$ such that*

$$\|\lambda \widetilde{\mathbf{R}}(\lambda)\|, \quad \left\|\lambda^{-1}\mathbf{A}\widetilde{\mathbf{R}}(\lambda)\right\|, \quad \left\|\mathbf{B}\widetilde{\mathbf{R}}(\lambda)\right\| \le M'_\varphi |\lambda|^{-1}, \quad \lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}. \tag{2.9}$$

Proof. We let

$$\begin{pmatrix} u_n \\ x_n \end{pmatrix}_{n\in N} \subset \mathcal{D}(\mathbf{B}), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix}, \quad \lim_{n\to\infty} \mathbf{B}\begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix}.$$

Then

$$\begin{cases} \lim_{n\to\infty} x_n = x, \\ \lim_{n\to\infty} B_1 x_n = y, \end{cases} \qquad \begin{cases} \lim_{n\to\infty} u_n = u, \\ \lim_{n\to\infty} B u_n = v, \end{cases}$$

and $\{P_1 u_n\}_{n\in N}$ is a Cauchy sequence in X/X_0 since

$$x_n \in P_1 u_n \quad \text{and} \quad \|P_1(u_n - u_m)\| \le \|x_n - x_m\|, \quad m, n \in N.$$

This combined with the closedness of B_1 and the completeness of $[\mathcal{D}(B)]_{P_1}$ indicates that

$$x \in \mathcal{D}(B_1), \quad u \in \mathcal{D}(B), \quad B_1 x = y, \quad Bu = v, \quad \lim_{n\to\infty} P_1 u_n = P_1 u.$$

We observe that

$$\mathrm{dist}(x_n,\ P_1 u) = \|P_1 u_n - P_1 u\|_{X/X_1},$$

because of $x_n \in P_1 u_n$. It follows that

$$\mathrm{dist}(x,\ P_1 u) = \lim_{n\to\infty} \mathrm{dist}(x_n,\ P_1 u) = 0,$$

and therefore $x \in P_1 u$. Thus we know that $\mathbf{B}$ is closed. A similar and simpler argument shows the closedness of $\mathbf{A}$.

Next, we observe that

$$\left\|\begin{pmatrix} u \\ x \end{pmatrix}\right\|_{[\mathcal{D}(\mathbf{A})]} = \|u\| + \|x\| + \|Au\| + \|A_1 x\| \quad \text{for} \quad \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}),$$

$$\left\|\begin{pmatrix} u \\ x \end{pmatrix}\right\|_{[\mathcal{D}(\mathbf{B})]} = \|u\| + \|x\| + \|Bu\| + \|B_1 x\| \quad \text{for} \quad \begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{B}),$$

and that $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A})$ (resp. $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{B})$) implies $x = Pu$ (resp. $x \in P_1 u$). From this and (2.5) – (2.7), we see easily that (2.8) is true.

Now, fix $\varphi \in (0,\theta)$ and let $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$. If $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$, then by Lemma 2.1 (2),

$$(\lambda^2 + A + \lambda B)D_\lambda x = 0,$$
$$u - D_\lambda x \in (\ker P) \cap \mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(A_0) \cap \mathcal{D}(B_0);$$

so

$$\begin{aligned}(\lambda^2 + \mathbf{A} + \lambda\mathbf{B})\begin{pmatrix} u \\ x \end{pmatrix} &= \begin{pmatrix} \lambda^2 + A + \lambda B & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1) \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} \\ &= \begin{pmatrix} \lambda^2 + A + \lambda B & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix}\begin{pmatrix} u - D_\lambda x \\ x \end{pmatrix} \\ &= \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix}\begin{pmatrix} u - D_\lambda x \\ x \end{pmatrix} \\ &= \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix}\begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix}.\end{aligned}$$

We then get

$$\lambda^2 + \mathbf{A} + \lambda\mathbf{B} = \begin{pmatrix} \lambda^2 + A_0 + \lambda B_0 & 0 \\ 0 & \lambda^2 + A_1 + \lambda B_1 \end{pmatrix}\begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix},$$

noting that

$$\begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A_0) \cap \mathcal{D}(B_0)) \times (\mathcal{D}(A_1) \cap \mathcal{D}(B_1))$$

implies $\begin{pmatrix} u \\ x \end{pmatrix} \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$. It follows that $\lambda^2 + \mathbf{A} + \lambda\mathbf{B}$ is invertible and

$$\begin{aligned}\mathbf{R}(\lambda) &:= \left(\lambda^2 + \mathbf{A} + \lambda\mathbf{B}\right)^{-1} \\ &= \begin{pmatrix} I & D_\lambda \\ 0 & I \end{pmatrix}\begin{pmatrix} R_0(\lambda) & 0 \\ 0 & R_1(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} R_0(\lambda) & D_\lambda R_1(\lambda) \\ 0 & R_1(\lambda) \end{pmatrix},\end{aligned} \tag{2.10}$$

$$\mathbf{AR}(\lambda) = \begin{pmatrix} A_0 R_0(\lambda) & A D_\lambda R_1(\lambda) \\ 0 & A_1 R_1(\lambda) \end{pmatrix}. \tag{2.11}$$

Take $\mu \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$. Then

$$AD_\mu, \quad BD_\mu \in \mathcal{L}(X, E), \tag{2.12}$$

by Lemma 2.1 (2). Using (2.12) and (H_2), we get from (2.1),

$$\sup\left\{\|D_\lambda\| + \left\|\lambda^{-2} A D_\lambda\right\|;\ \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} < \infty.$$

This combined with (H_2) and (H_3) yields that

$$\|\lambda\mathbf{R}(\lambda)\|,\ \left\|\lambda^{-1}\mathbf{AR}(\lambda)\right\| \le M'|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}, \tag{2.13}$$

for some constant $M' > M_\varphi$. From (2.10) we have

$$\widetilde{\mathbf{A}}\mathbf{R}(\lambda) = \begin{pmatrix} \widetilde{A}R_0(\lambda) & \widetilde{A}D_\lambda R_1(\lambda) \\ -G_0R_0(\lambda) & -G_0D_\lambda R_1(\lambda) + \widetilde{A}_1R_1(\lambda) \end{pmatrix}, \tag{2.14}$$

$$\widetilde{\mathbf{B}}\mathbf{R}(\lambda) = \begin{pmatrix} \widetilde{B}R_0(\lambda) & \widetilde{B}D_\lambda R_1(\lambda) \\ -G_1R_0(\lambda) & -G_1D_\lambda R_1(\lambda) + \widetilde{B}_1R_1(\lambda) \end{pmatrix}. \tag{2.15}$$

Since $\widetilde{A}$ (resp. $\widetilde{B}$) has A_0-bound (resp. B_0-bound) zero, there exists $a(\delta) > 0$, for each $\delta > 0$, such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\begin{aligned}
\left\|\widetilde{A}R_0(\lambda)\right\| &\le \delta\|A_0R_0(\lambda)\| + a(\delta)\|R_0(\lambda)\| \\
&\le \delta \sup\left\{\|A_0R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} \\
&\qquad + a(\delta)\sup\left\{\|\lambda^2R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\}|\lambda|^{-2}, \\
\left\|\lambda\widetilde{B}R_0(\lambda)\right\| &\le \delta\|\lambda B_0R_0(\lambda)\| + a(\delta)\|\lambda R_0(\lambda)\| \\
&\le \delta \sup\left\{\|\lambda B_0R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\} \\
&\qquad + a(\delta)\sup\left\{\|\lambda^2R_0(\lambda)\|;\quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}\right\}|\lambda|^{-1}.
\end{aligned}$$

Recalling (H_2), which implies

$$\|B_0R_0(\lambda)\| \le (1 + 2M_\varphi)|\lambda|^{-1}, \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

we see that the above supremums are all finite. Hence, for each $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\widetilde{A}R_0(\lambda)\right\|, \quad \left\|\lambda\widetilde{B}R_0(\lambda)\right\| \le \varepsilon + \beta(\varepsilon)|\lambda|^{-1}.$$

The same is true of each of $\|G_0R_0(\lambda)\|$, $\|\lambda G_1R_0(\lambda)\|$, $\left\|\widetilde{A}_1R_1(\lambda)\right\|$, $\left\|\lambda\widetilde{B}_1R_1(\lambda)\right\|$.

Note that

$$\widetilde{A}D_\mu, \quad \widetilde{B}D_\mu \in \mathcal{L}(X, E), \tag{2.16}$$

by (2.5) and Lemma 2.1 (2). We deduce from (2.1), (2.12) and (2.16) that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\begin{aligned}
\left\|\widetilde{A}D_\lambda R_1(\lambda)\right\| &\le \left\|\widetilde{A}D_\mu\right\|\|R_1(\lambda)\| + \left\|\widetilde{A}R_0(\lambda)\right\|\|D_\mu\|\left\|(\mu^2 - \lambda^2)R_1(\lambda)\right\| \\
&\qquad + \left\|\widetilde{A}R_0(\lambda)\right\|\|BD_\mu\|\|(\mu - \lambda)R_1(\lambda)\|, \\
\left\|\lambda\widetilde{B}D_\lambda R_1(\lambda)\right\| &\le \left\|\widetilde{B}D_\mu\right\|\|\lambda R_1(\lambda)\| + \left\|\lambda\widetilde{B}R_0(\lambda)\right\|\|D_\mu\|\left\|(\mu^2 - \lambda^2)R_1(\lambda)\right\| \\
&\qquad + \left\|\lambda\widetilde{B}R_0(\lambda)\right\|\|BD_\mu\|\|(\mu - \lambda)R_1(\lambda)\|.
\end{aligned}$$

Then, by (H_3) there is a constant $C_0 > 0$ such that for $\lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\widetilde{A}D_\lambda R_1(\lambda)\right\| \le C_0\left(|\lambda|^{-2} + \left\|\widetilde{A}R_0(\lambda)\right\|\right),$$

$$\left\|\lambda\widetilde{B}D_\lambda R_1(\lambda)\right\| \le C_0\left(|\lambda|^{-1} + \left\|\lambda\widetilde{B}R_0(\lambda)\right\|\right).$$

Similarly, we have

$$\|G_0 D_\lambda R_1(\lambda)\| \le C_1 \left(|\lambda|^{-2} + \|G_0 R_0(\lambda)\|\right), \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

$$\|\lambda G_1 D_\lambda R_1(\lambda)\| \le C_1 \left(|\lambda|^{-1} + \|\lambda G_1 R_0(\lambda)\|\right) \quad \lambda \in \omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

for some constant $C_1 > 0$.

The above arguments imply the existence of a constant $\omega'_\varphi > \omega_\varphi$ such that

$$\left\|\widetilde{\mathbf{A}}\mathbf{R}(\lambda)\right\| + \left\|\lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right\| \le \frac{1}{2}, \quad \lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi},$$

by the use of (2.14) and (2.15). Accordingly, we see that for $\lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\lambda^2 + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right) + \lambda\left(\mathbf{B} + \widetilde{\mathbf{B}}\right) = \left[I + \widetilde{\mathbf{A}}\mathbf{R}(\lambda) + \lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right](\lambda^2 + \mathbf{A} + \lambda\mathbf{B})$$

is invertible, and

$$\widetilde{\mathbf{R}}(\lambda) = \mathbf{R}(\lambda)\left[I + \widetilde{\mathbf{A}}\mathbf{R}(\lambda) + \lambda\widetilde{\mathbf{B}}\mathbf{R}(\lambda)\right]^{-1}.$$

This, together with (2.13), yields that for $\lambda \in \omega'_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$,

$$\left\|\lambda\widetilde{\mathbf{R}}(\lambda)\right\|, \left\|\lambda^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{R}}(\lambda)\right\| \le 2M'|\lambda|^{-1},$$

$$\left\|\widetilde{\mathbf{B}}\widetilde{\mathbf{R}}(\lambda)\right\| \le \left\|\lambda^{-1} - \lambda\widetilde{\mathbf{R}}(\lambda) - \lambda^{-1}\widetilde{\mathbf{A}}\widetilde{\mathbf{R}}(\lambda)\right\| \le (1 + 2M')|\lambda|^{-1}.$$

The proof is now complete.

By virtue of Theorem 2.3, we can obtain a fundamental solution operator of (1.8) as below.

Theorem 2.4. *Assume that the conditions of Theorem 2.3 hold. Define*

$$\widetilde{\mathbf{S}}(0) = 0, \quad \widetilde{\mathbf{S}}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\widetilde{\mathbf{R}}(\lambda)d\lambda \quad (t > 0), \tag{2.17}$$

where Γ *is any piecewise smooth curve in* $\omega_\varphi + \Sigma_{\frac{\pi}{2}+\varphi}$ $(\varphi \in (0,\theta))$ *going from* $\omega_\varphi + \infty e^{-i\delta}$ *to* $\omega_\varphi + \infty e^{i\delta}$ *(for some* $\delta \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \varphi\right)$*), and leaving* ω_φ *on its left. Then*

(1) *The operator function* $\widetilde{\mathbf{S}}(\cdot)$ *can be extended analytically to* Σ_θ *such that*

$$\widetilde{\mathbf{S}}(z)y \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}) \quad \text{for } y \in Y,\ z \in \Sigma_\theta,$$

and $\mathbf{A}\widetilde{\mathbf{S}}(\cdot)$, $\mathbf{B}\widetilde{\mathbf{S}}(\cdot)$ *are analytic in* Σ_θ*;*

(2) *for any* $\varphi \in (0,\theta)$, $\widetilde{\mathbf{S}}(\cdot)$ *is strongly continuous in* $\overline{\Sigma}_\varphi$*;*

(3) *for each* $y \in \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}$,

$$\lim_{t\to 0^+}\widetilde{\mathbf{S}}'(t)y = y, \quad \lim_{t\to 0^+}\mathbf{B}\widetilde{\mathbf{S}}(t)y = 0, \quad \lim_{t\to 0^+}\mathbf{A}\int_0^t \widetilde{\mathbf{S}}(s)yds = 0; \tag{2.18}$$

(4) *for each* $\varphi \in (0,\theta)$*, there exists* $M'_\varphi > 0$ *such that*

$$\left\|\widetilde{\mathbf{S}}'(z)\right\|, \left\|\mathbf{B}\widetilde{\mathbf{S}}(z)\right\|, \left\|\mathbf{A}\int_0^z \widetilde{\mathbf{S}}(\tau)d\tau\right\| \le M'_\varphi e^{\omega_\varphi \mathrm{Re} z}, \quad \text{for } z \in \Sigma_\varphi; \tag{2.19}$$

(5) *for any* $k \in \{0,1,2,3,4\}$, *there exist* M, $\omega > 0$ *such that*

$$\left\|\widetilde{\mathbf{S}}^{(k)}(t)\right\|,\ \left\|\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t)\right\|,\ \left\|\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t)\right\| \le Mt^{-(k-1)}e^{\omega t}, \quad t>0, \tag{2.20}$$

where

$$\widetilde{\mathbf{S}}^{(-i)}(t) := \int_0^t (t-s)^{i-1}\widetilde{\mathbf{S}}(s)ds, \quad i=1,2;$$

(6) *for every* $z \in \Sigma_\theta$,

$$\widetilde{\mathbf{S}}''(z) + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)\widetilde{\mathbf{S}}'(z) + \left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\widetilde{\mathbf{S}}(z) = 0, \tag{2.21}$$

$$\widetilde{\mathbf{S}}''(z)y + \widetilde{\mathbf{S}}'(z)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y + \widetilde{\mathbf{S}}(z)\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y = 0, \quad y \in \mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B}). \tag{2.22}$$

Proof. By means of Theorem 2.3, the arguments similar to those in the proof of the implication (ii) $\Longrightarrow$ (i) of [24, Theorem 1.1, Section 4.1] justify assertions (1) - (4) and (6). In order to show assertion (5), we choose $\Gamma = \omega_\varphi + \Gamma_1$ with

$$\Gamma_1 := \left\{\rho e^{\pm i\frac{\pi+\varphi}{2}};\quad \rho \ge 1\right\} \cup \left\{e^{i\theta}; |\theta| \le \delta\right\}.$$

From (2.17) we get

$$\begin{aligned}
\widetilde{\mathbf{S}}^{(k)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^k e^{\lambda t}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{t\Gamma_1}\left(t^{-1}\mu+\omega_\varphi\right)^k e^{\mu}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1}\left(t^{-1}\mu+\omega_\varphi\right)^k e^{\mu}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu, \\
\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{k-1} e^{\lambda t}\mathbf{B}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1}\left(t^{-1}\mu+\omega_\varphi\right)^{k-1} e^{\mu}\mathbf{B}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu, \\
\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{k-2} e^{\lambda t}\mathbf{A}\widetilde{\mathbf{R}}(\lambda)d\lambda \\
&= \frac{t^{-1}e^{\omega_\varphi t}}{2\pi i}\int_{\Gamma_1}\left(t^{-1}\mu+\omega_\varphi\right)^{k-2} e^{\mu}\mathbf{A}\widetilde{\mathbf{R}}(t^{-1}\mu+\omega_\varphi)d\mu.
\end{aligned}$$

Therefore, using Theorem 2.3 yields that for $t > 0$,

$$\begin{aligned}
&\left\|\widetilde{\mathbf{S}}^{(k)}(t)\right\|,\ \left\|\mathbf{B}\widetilde{\mathbf{S}}^{(k-1)}(t)\right\|,\ \left\|\mathbf{A}\widetilde{\mathbf{S}}^{(k-2)}(t)\right\| \\
\le\ & \frac{1}{2\pi}t^{-(k-1)}e^{\omega_\varphi t}\int_\Gamma \left|\mu+t^2\right|^{k-2} e^{\mathrm{Re}\mu}|d\mu| \\
\le\ & \text{const}\ t^{-(k-1)}\left(1+t^2\right)e^{\omega_\varphi t}.
\end{aligned}$$

The proof is then complete.

3. The main theorem for problem (1.8)

Definition 3.1. Assume that $\mathbf{A}$, $\mathbf{B}$ are closed, and $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$ satisfy (2.8). Let $h \in C([0,T];\mathbf{E})$.

(i) A function $y(\cdot)$ is called a classical solution of (1.8) if $y(\cdot) \in C^2((0,T];\mathbf{E}) \cap C^1([0,T];\mathbf{E})$,

$$y(\cdot) \in C((0,T];[\mathcal{D}(\mathbf{A})]), \quad \int_0^{\cdot} y(\sigma)d\sigma \in C([0,T];[\mathcal{D}(\mathbf{A})]),$$

$$y'(\cdot) \in C((0,T];[\mathcal{D}(\mathbf{B})]), \quad y(\cdot) - y(0) \in C([0,T];[\mathcal{D}(\mathbf{B})]),$$

and (1.8) is satisfied.

(ii) A function $y(\cdot)$ is called a strict solution of (1.8) if $y(\cdot) \in C^2([0,T];\mathbf{E}) \cap C([0,T];[\mathcal{D}(\mathbf{A})])$, $y'(\cdot) \in C([0,T];[\mathcal{D}(\mathbf{B})])$, and (1.8) is satisfied.

Remark 3.2. It can be seen from (2.8) that

(1) if $y(\cdot)$ is a classical solution of (1.8), then

$$\widetilde{\mathbf{B}}y'(\cdot), \quad \widetilde{\mathbf{A}}y(\cdot) \in C((0,T];\mathbf{E}),$$
$$\widetilde{\mathbf{B}}(y(\cdot) - y(0)), \quad \widetilde{\mathbf{A}}\int_0^{\cdot} y(\sigma)d\sigma \in C([0,T];\mathbf{E});$$

(2) if $y(\cdot)$ is a strict solution of (1.8), then

$$\widetilde{\mathbf{B}}y'(\cdot), \quad \widetilde{\mathbf{A}}y(\cdot) \in C([0,T];\mathbf{E}).$$

Now we introduce a subset $\boldsymbol{\Upsilon}$ of $\mathbf{E}$, which is closely related to the Brezis-Fraenkel condition in [3] (see also [14, Appendix], [19]). Put

$$\boldsymbol{\Upsilon} := \left\{ y \in \mathcal{D}(\mathbf{B}); \quad \lim_{t\to 0^+} \Psi(t,y) = 0 \right\}, \tag{3.1}$$

where

$$\Psi(t,y) := \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left(t\|v\|_{[\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})]} + \|y-v\|_{[\mathcal{D}(\mathbf{B})]} + t^{-1}\|y-v\| \right), \quad t \in (0,T], \ y \in \mathcal{D}(\mathbf{B}). \tag{3.2}$$

It is not difficult to see that

$$\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}) \subset \boldsymbol{\Upsilon} \subset \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}.$$

We are now in a position to present our main theorem.

Theorem 3.3. *Let the hypotheses of Theorem 2.3 hold, $h \in C^\alpha([0,T];\mathbf{E})$ ($\alpha \in (0,1)$), $y_0 \in \mathcal{D}(\mathbf{A}) \cup \boldsymbol{\Upsilon}$, and $y_1 \in \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}$. Then*

(1) *problem (1.8) has a unique classical solution $y(\cdot)$, given by*

$$y(t) = \widetilde{\mathbf{C}}(t)y_0 + \widetilde{\mathbf{S}}(t)y_1 + \int_0^t \widetilde{\mathbf{S}}(t-s)h(s)ds, \quad t \in [0,T], \tag{3.3}$$

where for $t \in [0,T]$,

$$\widetilde{\mathbf{C}}(t)y_0 := \begin{cases} y_0 - \int_0^t \widetilde{\mathbf{S}}(s)(\mathbf{A}+\widetilde{\mathbf{A}})y_0 ds, & if \quad y_0 \in \mathcal{D}(\mathbf{A}), \\ \left(\widetilde{\mathbf{S}}'(t) + \widetilde{\mathbf{S}}(t)(\mathbf{B}+\widetilde{\mathbf{B}})\right) y_0 & if \quad y_0 \in \boldsymbol{\Upsilon}. \end{cases} \tag{3.4}$$

(2) *the function $y(\cdot)$ satisfies the following regularity property and estimates:*

$$y''(\cdot),\ \mathbf{B}y'(\cdot),\ \mathbf{A}y(\cdot) \in C^\alpha([\varepsilon, T]; \mathbf{E}), \quad \varepsilon \in (0, T); \tag{3.5}$$

$$\|y(t)\| \leq \text{const} \left(\|h\|_{C([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{B})]} + \|y_1\|\right), \quad \text{if } y_0 \in \mathbf{\Upsilon},\ t \in [0, T]; \tag{3.6}$$

$$\|y'(t)\| \leq \text{const} \left(\|h\|_{C([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{A})]} + \|y_1\|\right), \quad \text{if } y_0 \in \mathcal{D}(\mathbf{A}),\ t \in [0, T]; \tag{3.7}$$

$$\begin{aligned} &\|y''(t)\| + \|y'(t)\|_{[\mathcal{D}(\mathbf{B})]} + \|y(t)\|_{[\mathcal{D}(\mathbf{A})]} \\ &\leq \text{const} \left(\|h\|_{C^\alpha([0,T];\mathbf{E})} + \|y_0\|_{[\mathcal{D}(\mathbf{A})]} + \|y_1\|_{[\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})]}\right), \\ &\qquad \text{if } y_0 \in \mathcal{D}(\mathbf{A}),\ y_1 \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B}),\ t \in (0, T]. \end{aligned} \tag{3.8}$$

(3) *the function $y(t)$ is a strict solution of (1.8) provided $y_0 \in \mathcal{D}(\mathbf{A})$, $y_1 \in \mathbf{\Upsilon}$, and*

$$(\mathbf{A} + \widetilde{\mathbf{A}})y_0 + (\mathbf{B} + \widetilde{\mathbf{B}})y_1 - h(0) \in \overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})}. \tag{3.9}$$

Proof. We will use freely the closedness of $\mathbf{A}$, $\mathbf{B}$ and the fact (2.8) concerning $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$. Put

$$y_*(t) := \int_0^t \widetilde{\mathbf{S}}(t-s)h(s)ds, \quad t \in [0, T].$$

We then have (noting $\widetilde{\mathbf{S}}(0) = 0$)

$$y_*(t) = \int_0^t \widetilde{\mathbf{S}}(\sigma)h(t)d\sigma + \int_0^t \widetilde{\mathbf{S}}(t-\sigma)(h(\sigma) - h(t))d\sigma, \quad t \in [0, T], \tag{3.10}$$

$$y_*'(t) = \widetilde{\mathbf{S}}(t)h(t) + \int_0^t \widetilde{\mathbf{S}}'(t-\sigma)(h(\sigma) - h(t))d\sigma, \quad t \in [0, T], \tag{3.11}$$

$$y_*''(t) = \widetilde{\mathbf{S}}'(t)h(t) + \int_0^t \widetilde{\mathbf{S}}''(t-\sigma)(h(\sigma) - h(t))d\sigma, \quad t \in (0, T], \tag{3.12}$$

in view of the estimates

$$\|h(\sigma) - h(t)\| \leq \text{const } (t-\sigma)^\alpha, \quad 0 \leq \sigma \leq t \leq T, \tag{3.13}$$

and (2.20). Thus, we infer by (3.13), (2.20), (2.21) and Theorem 2.4 (1) and (2) that

$$y_*''(\cdot), \quad \mathbf{B}y_*'(\cdot), \quad \mathbf{A}y_*(\cdot) \in C((0, T]; \mathbf{E}), \tag{3.14}$$

$$y_*'(\cdot), \quad \mathbf{B}y_*(\cdot), \quad \mathbf{A}\int_0^\cdot y_*(\sigma)d\sigma \in C([0, T]; \mathbf{E}), \tag{3.15}$$

$$y_*''(t) + \left(\mathbf{B} + \widetilde{\mathbf{B}}\right)y_*'(t) + \left(\mathbf{A} + \widetilde{\mathbf{A}}\right)y_*(t) = h(t), \quad t \in (0, T]. \tag{3.16}$$

Clearly

$$y_*(0) = 0, \quad y_*'(0) = 0, \tag{3.17}$$

by (3.10), (3.11) and (2.17). Next, we fix $\varepsilon \in (0,T)$. Using (3.12), (3.13) and (2.20) yields that for $\varepsilon \le s < t \le T$,

$$\begin{aligned}
&\|y_*''(t) - y_*''(s)\| \\
\le\ & \left\|\widetilde{\mathbf{S}}'(t)\right\| \|h(t)-h(s)\| + \left\|\int_s^t \widetilde{\mathbf{S}}''(\sigma)d\sigma\right\| \|h(s)\| \\
&\quad + \int_0^s \left\|\widetilde{\mathbf{S}}''(t-\sigma) - \widetilde{\mathbf{S}}''(s-\sigma)\right\| \|h(\sigma)-h(s)\|d\sigma \\
&\quad + \left\|\int_0^s \widetilde{\mathbf{S}}''(t-\sigma)d\sigma\right\| \|h(s)-h(t)\| \\
&\quad + \int_s^t \left\|\widetilde{\mathbf{S}}''(t-\sigma)\right\| \|h(\sigma)-h(t)\|d\sigma \\
\le\ & \text{const}\ \left[(t-s)^\alpha + \int_s^t \sigma^{-1}d\sigma + \int_0^s \left|\int_{s-\sigma}^{t-\sigma} \tau^{-2}d\tau\right| (s-\sigma)^\alpha d\sigma \right. \\
&\quad \left. + \left\|\widetilde{\mathbf{S}}(t-s) - \widetilde{\mathbf{S}}(t)\right\| (t-s)^\alpha + \int_s^t (t-\sigma)^{\alpha-1}d\sigma\right] \\
\le\ & \text{const}\ \left[(t-s)^\alpha + \varepsilon^{-1}(t-s) + (t-s)\int_0^s (t-\sigma)^{-1}(s-\sigma)^{\alpha-1}d\sigma\right] \\
\le\ & \text{const}\ (t-s)^\alpha.
\end{aligned}$$

In a similar way, we obtain from (3.10) and (3.11)

$$\|\mathbf{B}y_*'(t) - \mathbf{B}y_*'(s)\|,\ \|\mathbf{A}y_*(t) - \mathbf{A}y_*(s)\| \le \text{const}\ (t-s)^\alpha, \quad \varepsilon \le s < t \le T.$$

Therefore

$$y_*''(\cdot),\ \mathbf{B}y_*'(\cdot),\ \mathbf{A}y_*(\cdot) \in C^\alpha([\varepsilon,T];\mathbf{E}), \quad \varepsilon \in (0,T). \tag{3.18}$$

We now take care of $\widetilde{\mathbf{C}}(\cdot)y_0$ and $\widetilde{\mathbf{S}}(\cdot)y_1$. By (3.4) and the related properties of $\widetilde{\mathbf{S}}(\cdot)$ (see Theorem 2.4), we get

$$\widetilde{\mathbf{C}}(0)y_0 = y_0,\ \ \widetilde{\mathbf{S}}(0)y_1 = 0,\ \ \widetilde{\mathbf{C}}'(0)y_0 = 0,\ \ \widetilde{\mathbf{S}}'(0)y_1 = y_1, \tag{3.19}$$

$$\widetilde{\mathbf{C}}''(\cdot)y_0,\ \mathbf{B}\widetilde{\mathbf{C}}'(\cdot)y_0,\ \mathbf{A}\widetilde{\mathbf{C}}(\cdot)y_0 \in C((0,T];\mathbf{E}), \tag{3.20}$$

$$\widetilde{\mathbf{S}}''(\cdot)y_1,\ \mathbf{B}\widetilde{\mathbf{S}}'(\cdot)y_1,\ \mathbf{A}\widetilde{\mathbf{S}}(\cdot)y_1 \in C((0,T];\mathbf{E}), \tag{3.21}$$

and

$$\begin{aligned}
&\widetilde{\mathbf{C}}''(t)y_0 + \widetilde{\mathbf{S}}''(t)y_1 + \left(\mathbf{B}+\widetilde{\mathbf{B}}\right)\left(\widetilde{\mathbf{C}}'(t)y_0 + \widetilde{\mathbf{S}}'(t)y_1\right) \\
&\qquad + \left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\left(\widetilde{\mathbf{C}}(t)y_0 + \widetilde{\mathbf{S}}(t)y_1\right) \\
=\ & 0, \quad t \in (0,T].
\end{aligned} \tag{3.22}$$

Moreover, using (2.20), we see easily that for $\varepsilon \le s < t \le T$,

$$\left.\begin{array}{l} \left\|\widetilde{\mathbf{C}}''(t)y_0 - \widetilde{\mathbf{C}}''(s)y_0\right\| \\ \left\|\mathbf{B}\widetilde{\mathbf{C}}'(t)y_0 - \mathbf{B}\widetilde{\mathbf{C}}'(s)y_0\right\| \\ \left\|\mathbf{A}\widetilde{\mathbf{C}}(t)y_0 - \mathbf{A}\widetilde{\mathbf{C}}(s)y_0\right\| \end{array}\right\} \le \text{const } (t-s), \tag{3.23}$$

$$\left.\begin{array}{l} \left\|\widetilde{\mathbf{S}}''(t)y_1 - \widetilde{\mathbf{S}}''(s)y_1\right\| \\ \left\|\mathbf{B}\widetilde{\mathbf{S}}'(t)y_1 - \mathbf{B}\widetilde{\mathbf{S}}'(s)y_1\right\| \\ \left\|\mathbf{A}\widetilde{\mathbf{S}}(t)y_1 - \mathbf{A}\widetilde{\mathbf{S}}(s)y_1\right\| \end{array}\right\} \le \text{const } (t-s). \tag{3.24}$$

In the following, we will show that

$$\widetilde{\mathbf{C}}'(t)y_0,\ \mathbf{B}\left(\widetilde{\mathbf{C}}(t)y_0 - y_0\right),\ \mathbf{A}\int_0^t \widetilde{\mathbf{C}}(\sigma)y_0 d\sigma \longrightarrow 0 \tag{3.25}$$

as $t \to 0^+$. When $y_0 \in \mathcal{D}(\mathbf{A})$, (3.25) follows immediately from (3.4) and Theorem 2.4 (2) and (4). Now let $y_0 \in \boldsymbol{\Upsilon}$. Making use of (2.20), (2.22) and noting $\widetilde{\mathbf{S}}(0) = 0$, $\widetilde{\mathbf{S}}'(0)v = v$ for $v \in \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})$, we obtain

$$\begin{aligned} & \left\|\widetilde{\mathbf{C}}'(t)y_0\right\| \\ = \ & \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left\|\widetilde{\mathbf{C}}'(t)(y_0 - v) - \widetilde{\mathbf{S}}(t)\left(\mathbf{A} + \widetilde{\mathbf{A}}\right)v\right\| \\ \le \ & \text{const} \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left(t^{-1}\|y_0 - v\| + \left\|\left(\mathbf{B} + \widetilde{\mathbf{B}}\right)(y_0 - v)\right\| + t\left\|\left(\mathbf{A} + \widetilde{\mathbf{A}}\right)v\right\|\right) \\ \le \ & \text{const } \Psi(t, y_0), \quad t \in (0, T] \qquad \text{(by (3.2))}, \end{aligned}$$

$$\begin{aligned} & \left\|\mathbf{B}\left(\widetilde{\mathbf{C}}(t)y_0 - y_0\right)\right\| \\ = \ & \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left\|\mathbf{B}\widetilde{\mathbf{C}}(t)(y_0 - v) - \mathbf{B}\int_0^t \widetilde{\mathbf{S}}(\sigma)(\mathbf{A} + \widetilde{\mathbf{A}})v d\sigma + \mathbf{B}(v - y_0)\right\| \\ \le \ & \text{const } \Psi(t, y_0), \quad t \in (0, T], \end{aligned}$$

$$\begin{aligned} & \left\|\mathbf{A}\int_0^t \widetilde{\mathbf{C}}(\sigma)y_0 d\sigma\right\| \\ = \ & \inf_{v\in\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} \left\|\mathbf{A}\left(\widetilde{\mathbf{S}}(t) + \int_0^t \widetilde{\mathbf{S}}(\sigma)\left(\mathbf{B} + \widetilde{\mathbf{B}}\right)d\sigma\right)(y_0 - v)\right. \\ & \qquad \left. - \mathbf{A}\int_0^t (t - \sigma)\widetilde{\mathbf{S}}(\sigma)\left(\mathbf{A} + \widetilde{\mathbf{A}}\right)v d\sigma + t\mathbf{A}v\right\| \\ \le \ & \text{const } \Psi(t, y_0), \quad t \in (0, T]. \end{aligned}$$

This leads to (3.25) in view of the definition of $\boldsymbol{\Upsilon}$ (see (3.1)). Combining (2.18), (3.14) – (3.17), (3.19) – (3.22), and (3.25) together, we deduce that the function $y(\cdot)$ defined by (3.3) is a classical solution of problem (1.8).

In order to show the uniqueness, let $v(\cdot)$ be another classical solution of (1.8). Then

$$v'(t)-y'(t)+\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)(v(t)-y(t))+\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\int_0^t(v(s)-y(s))ds=0,\quad t\in[0,T].$$

So a calculation involving integration by parts shows that for $t\in[0,T]$, λ large enough,

$$\begin{aligned}&\left(\lambda+\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)+\lambda^{-1}\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\right)\int_0^t e^{\lambda(t-s)}(v(s)-y(s))ds\\ =\ &-v(t)+y(t)+\lambda^{-1}\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)\int_0^t(v(\sigma)-y(\sigma))d\sigma.\end{aligned}$$

Hence for $t\in[0,T]$,

$$\lim_{\lambda\to\infty}e^{-\lambda}\int_0^t e^{\lambda(t-s)}(v(s)-y(s))ds=0$$

since $\lim_{\lambda\to\infty}\lambda^2e^{-\lambda}\widetilde{\mathbf{R}}(\lambda)w=0$ ($w\in\mathbf{E}$). This yields that $v(t)=y(t)$ for all $t\in[0,T]$, in view of [20, Lemma 1.1, p. 100]. Therefore, assertion (1) is valid. The regularity property (3.5) comes from (3.18), (3.23) and (3.24). Based on the expression (3.3) of $y(t)$, we derive the estimates (3.6) – (3.8) by (2.19) and (2.21).

Finally, assume $y_0\in\mathcal{D}(\mathbf{A})$ and $y_1\in\boldsymbol{\Upsilon}$ satisfying (3.9). To prove that $y(\cdot)$ (in this case) is a strict solution, we observe by (3.3), (2.22) and (3.12) that

$$\begin{aligned}y''(t)\ =\ &-\widetilde{\mathbf{S}}'(t)\left(\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y_0+\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1-h(0)\right)\\ &+\widetilde{\mathbf{S}}''(t)y_1+\widetilde{\mathbf{S}}'(t)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1+\widetilde{\mathbf{S}}'(t)(h(t)-h(0))\\ &+\int_0^t\widetilde{\mathbf{S}}''(t-\sigma)(h(\sigma)-h(t))d\sigma,\quad t\in(0,T].\end{aligned}$$

The same reasoning as for (3.25) (in the case of $y_0\in\boldsymbol{\Upsilon}$) gives that

$$\lim_{t\to0^+}\left(\widetilde{\mathbf{S}}''(t)y_1+\widetilde{\mathbf{S}}'(t)\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1\right)=0.$$

Therefore

$$\lim_{t\to0^+}y''(t)=-\left(\mathbf{A}+\widetilde{\mathbf{A}}\right)y_0-\left(\mathbf{B}+\widetilde{\mathbf{B}}\right)y_1+h(0),\tag{3.26}$$

by (2.18) – (2.20) and (3.13). Analogously, we obtain

$$\lim_{t\to0^+}\mathbf{B}y'(t)=\mathbf{B}y_1,\quad\lim_{t\to0^+}\mathbf{A}y(t)=\mathbf{A}y_0.\tag{3.27}$$

Thus, (3.26) and (3.27) together with assertion (1) justify assertion (3). This finishes the proof.

4. Examples

In this section, we present three examples, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Example 4.1. We consider a linear viscoelastic beam with a tip mass whose dynamic evolution is described by the following system:

$$(4.1)\qquad \begin{cases} \partial_t^2 u + \partial_\xi^4 u + \beta\partial_\xi^4\partial_t u = f, & t\in[0,T],\ \xi\in(0,1),\\ u(t,0) = \partial_\xi u(t,0) = 0, & t\in[0,T],\\ \kappa\partial_t^2 u(t,1) - \partial_\xi^3 u(t,1) - \beta\partial_\xi^3\partial_t u(t,1) = 0, & t\in[0,T],\\ \gamma\partial_t^2\partial_\xi u(t,1) + \partial_\xi^2 u(t,1) + \beta\partial_\xi^2\partial_t u(t,1) = 0, & t\in[0,T],\\ u(0,\xi) = \varphi_0(\xi),\quad \partial_t u(0,\xi) = \varphi_1(\xi), & \xi\in[0,1], \end{cases}$$

where β, κ and γ are positive constants, and $f\in C^\alpha([0,T];L^2(0,1))$ $(\alpha\in(0,1))$.

Take

$$E = L^2(0,1),\quad X = \mathbf{C}^2,$$

$$A = \frac{d^4}{d\xi^4}\quad\text{with}\quad \mathcal{D}(A) = \{\varphi\in H^4(0,1);\ \ \varphi(0) = \varphi'(0) = 0\},$$

$$P\varphi = \begin{pmatrix}\varphi(1)\\ \varphi'(1)\end{pmatrix}\quad\text{for}\quad \varphi\in\mathcal{D}(P) := \mathcal{D}(A),$$

$$G_0\varphi = \begin{pmatrix}\kappa^{-1}\varphi'''(1)\\ -\gamma^{-1}\varphi''(1)\end{pmatrix}\quad\text{for}\quad \varphi\in\mathcal{D}(G_0) := \mathcal{D}(A),$$

$$B = \beta A,\quad P_1 = P,\quad G_1 = \beta G_0,$$

$$A_1 = 0,\ \ B_1 = 0,\ \ \widetilde{A} = 0,\ \ \widetilde{A}_1 = 0,\ \ \widetilde{B} = 0,\ \ \widetilde{B}_1 = 0.$$

Then we see easily that $B_0 = \beta A_0$ (see (1.7)),

$$A_0 = \frac{d^4}{d\xi^4}\quad\text{with}\quad \mathcal{D}(A_0) = \{\varphi\in H^4(0,1);\ \ \varphi(0) = \varphi'(0) = \varphi(1) = \varphi'(1) = 0\},$$

and

$$\|\cdot\|_{A,P}\sim\|\cdot\|_{B,P_1}\sim\|\cdot\|_{H^4}.$$

Obviously (H_1), (H_3) and (2.6) hold. From the fact that $-B_0$ generates a strongly continuous analytic semigroup on $L^2(0,1)$, it follows that (H_2) holds too (cf. [24, Corollary 1.6, p. 149]). It is clear that $G_0 : [\mathcal{D}(A_0)]\to\mathbf{C}^2$ is bounded and so compact. By [7, Lemma 2.16, p. 179] (with a slight improvement), we deduce that G_0 is A_0-bounded with A_0-bound zero. Likewise, G_1 is B_0-bounded with B_0-bound zero. Moreover, we note that $\overline{\mathcal{D}(\mathbf{A})\cap\mathcal{D}(\mathbf{B})} = \mathbf{E}$ (see Section 1 for the definitions of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$), since $\mathcal{D}(A_0)\cap\mathcal{D}(B_0)$ is dense in $L^2(0,1)$ and $P(\mathcal{D}(A)\cap\mathcal{D}(B)) = X$. Setting

$$y(t) = \begin{pmatrix}u(t)\\ x(t)\end{pmatrix},\quad u(t) = u(t,\cdot),\quad x(t) = \begin{pmatrix}u(t,1)\\ \partial_\xi u(t,1)\end{pmatrix},\qquad t\in[0,T],$$

we apply Theorem 3.3 to conclude:

For every $\varphi_0,\ \varphi_1\in H^4(0,1)$ with $\varphi_i(0) = \varphi_i'(0) = 0$ $(i = 0,1)$, problem (4.1) has a unique solution

$$(4.2)\qquad u\in C^2\left([0,T];L^2(0,1)\right)\cap C^1([0,T];H^4(0,1));$$

moreover,

$$\partial_t^2 u \in C^\alpha\left([\varepsilon,T];L^2(0,1)\right), \quad \partial_t u \in C^\alpha\left([\varepsilon,T];H^4(0,1)\right), \qquad \varepsilon \in (0,T),$$

$$\begin{aligned}&\left\|\partial_t^2 u(t,\cdot)\right\|_{L^2(0,1)} + \|\partial_t u(t,\cdot)\|_{H^4(0,1)} \\ \le\ & \text{const}\ \left(\|f\|_{C^\alpha([0,T];L^2(0,1))} + \|\varphi_0\|_{H^4(0,1)} + \|\varphi_1\|_{H^4(0,1)}\right), \quad t \in [0,T].\end{aligned}$$

For the uniqueness, we have made the observation that if u is a solution of problem (4.1) satisfying (4.2), then

$$x'(t) = P_1 u'(t) \ \ (t \in [0,T]), \quad x(0) = \begin{pmatrix}\varphi_0(1)\\ \varphi_0'(1)\end{pmatrix}, \quad x'(0) = \begin{pmatrix}\varphi_1(1)\\ \varphi_1'(1)\end{pmatrix}.$$

Remark 4.2. In [1, Theorem 5.1], problem (4.1) was discussed in a much weaker form. The weak solution v obtained there satisfies that

$$v,\ v' \in L^2((0,T);\mathcal{E}), \quad (Qv)',\ (Qv)'' \in L^2((0,T);\mathcal{E}'),$$

and

$$\begin{cases} (Qv)'' + (Mv)' + Nv = f \quad \text{in}\ \ L^2((0,T);\mathcal{E}'), \\ Qv(0) = Q\varphi_0, \quad (Qv)'(0) = Q\varphi_1, \end{cases}$$

where $\mathcal{E} := \{\varphi \in H^2(0,1);\ \ \varphi(0) = \varphi'(0) = 0\}$, the dual space $\mathcal{E}' \supset L^2(0,1) \supset \mathcal{E}$, and the operators Q, M, $N : \mathcal{E} \to \mathcal{E}'$ are defined by

$$\langle Q\varphi,\ \psi\rangle = \int_0^1 \varphi\psi d\xi + \kappa\phi(1)\psi(1) + \gamma\phi'(1)\psi'(1),$$

$$\langle M\varphi,\ \psi\rangle = \int_0^1 \beta\varphi''\psi'' d\xi,$$

$$\langle N\varphi,\ \psi\rangle = \int_0^1 \varphi''\psi'' d\xi, \qquad \varphi,\psi \in \mathcal{E}.$$

Example 4.3. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, and let $\rho > 0$. We consider the mixed boundary control problem for a structurally damped plate-like equation:

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = 0, & \text{in } [0,T]\times\Omega, \\ \partial_t^2\left(u\big|_{\partial\Omega}\right) = w, & \text{in } [0,T]\times\partial\Omega, \\ \Delta u\big|_{\partial\Omega} = 0, & \text{in } [0,T]\times\partial\Omega, \\ u(0,\cdot) = \varphi_0, \quad \partial_t u(0,\cdot) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{4.3}$$

where w is the control force.

The objective is to show that problem (4.3) (with a suitable w) is well posed in $L^p(\Omega)$ $(1 < p < \infty)$.

We consider the case where w is built up by a feedback control law:

$$w = \langle \Delta u, a\rangle b + g$$

with

$$a \in L^q(\Omega)\ \left(\frac{1}{q} + \frac{1}{p} = 1\right), \quad b \in W^{2,p}(\partial\Omega),$$
$$g \in C^\alpha\left([0,T];W^{2,p}(\partial\Omega)\right)\ \ (\alpha \in (0,1)).$$

When $a = 0$, (4.3) becomes an open loop problem.

In order to apply our theorems, we take

$$E = L^p(\Omega), \quad X = W^{2-\frac{1}{p},p}(\partial\Omega),$$
$$B = -\rho\Delta \quad \text{with} \quad \mathcal{D}(B) = W^{2,p}(\Omega),$$
$$A = \Delta^2 \quad \text{with} \quad \mathcal{D}(A) = \{\varphi \in \mathcal{D}(B^2); \ \ \Delta\varphi\big|_{\partial\Omega} = 0\},$$
$$G_0\varphi = \langle \Delta\varphi, a\rangle b \quad \text{for} \quad \varphi \in \mathcal{D}(G_0) := \mathcal{D}(A),$$
$$P\varphi = \varphi\Big|_{\partial\Omega} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A), \quad P_1 = P,$$
$$A_1 = 0, \ \ B_1 = 0, \ \ \widetilde{A} = 0, \ \ \widetilde{A}_1 = 0, \ \ \widetilde{B} = 0, \ \ \widetilde{B}_1 = 0, \ \ G_1 = 0.$$

We claim that (H_1) is satisfied. In fact, a trace theorem [23, Section 5.5.2, pp. 390, 391] says that

$$\mathcal{P} : \varphi \longmapsto \big(\Delta\varphi, \ \varphi\big|_{\partial\Omega}\big)$$

is an isomorphic mapping from $W^{2,p}(\Omega)$ onto $L^p(\Omega) \times W^{2-\frac{1}{p},p}(\partial\Omega)$. Hence, given $x \in W^{2-\frac{1}{p},p}(\partial\Omega)$, there exists $\varphi \in W^{2,p}(\Omega)$ such that

$$\Delta\varphi = 0, \quad \varphi\big|_{\partial\Omega} = x.$$

It follows immediately that

$$\varphi \in \mathcal{D}(A) \quad \text{and} \quad P\varphi = x.$$

So $P(\mathcal{D}(A) \cap \mathcal{D}(B)) = X$. Next we show the completeness of $[\mathcal{D}(A)]_P$. To this end, we take a Cauchy sequence $\{\psi_n\}_{n\in N}$ in $[\mathcal{D}(A)]_P$. Then, there exist r, $r_0 \in L^p(\Omega)$ and $v \in W^{2-\frac{1}{p},p}(\partial\Omega)$ such that

$$\lim_{n\to\infty} \|\psi_n - r\|_{L^p(\Omega)} = 0, \tag{4.4}$$
$$\lim_{n\to\infty} \big\|\Delta^2\psi_n - r_0\big\|_{L^p(\Omega)} = 0, \tag{4.5}$$
$$\lim_{n\to\infty} \big\|\psi_n\big|_{\partial\Omega} - v\big\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} = 0, \tag{4.6}$$
$$\Delta\psi_n\big|_{\partial\Omega} = 0. \tag{4.7}$$

According to (4.5) and (4.7), the isomorphism $\mathcal{P}$ implies the existence of r_1, $r_2 \in W^{2,p}(\Omega)$ such that

$$\lim_{n\to\infty} \|\Delta\psi_n - r_1\|_{L^p(\Omega)} = 0, \tag{4.8}$$
$$\Delta r_1 = r_0, \quad r_1\big|_{\partial\Omega} = 0. \tag{4.9}$$

Using (4.4), (4.6) and (4.8) yields that

$$r \in W^{2,p}(\Omega), \quad \Delta r = r_1, \quad r\big|_{\partial\Omega} = v.$$

From this, (4.9) and (4.4) – (4.7), we deduce that

$$r \in \mathcal{D}(A) \quad \text{and} \quad \lim_{n\to\infty} \|\psi_n - r\|_{A,P} = 0.$$

Therefore $[\mathcal{D}(A)]_P$ is complete. The completeness of $[\mathcal{D}(B)]_{P_1}$ can be verified in the same way. Moreover, using the $\mathcal{P}$ again we find that

$$\|\cdot\|_{A,P} \sim \|\cdot\|_{W^{2,p}(\Omega)} + \|\Delta\cdot\|_{W^{2,p}(\Omega)}, \quad \|\cdot\|_{B,P_1} \sim \|\cdot\|_{W^{2,p}(\Omega)}. \tag{4.10}$$

Clearly $B_0 := B\big|_{\ker P_1} = -\rho\Delta_D$ and $A_0 := A\big|_{\ker P} = \Delta_D^2$ (Δ_D is the Dirichlet Laplacian). By [14, Theorem 3.4], (H$_2$) holds. The first equivalent relation in (4.10) tells us that $G_0 \in \mathcal{L}([\mathcal{D}(A)]_P, X)$. Obviously, G_0 is relatively Δ_D^2-bounded with Δ_D^2-bound zero. Thus the hypotheses of Theorem 2.3 are fulfilled. So Theorem 2.3 is applicable to this situation, in which $\overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} = \mathbf{E}$,

$$y(t) = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad u(t) = u(t,\cdot), \quad x(t) := u(t,\cdot)\big|_{\partial\Omega}, \quad t \in [0,T].$$

Noting (4.10), we then obtain the following conclusion:

For every $\varphi_0,\ \varphi_1 \in W^{2,p}(\Omega)$ with $\Delta\varphi_0,\ \Delta\varphi_1 \in W^{2,p}(\Omega)$ and $\Delta\varphi_0\big|_{\partial\Omega} = \Delta\varphi_1\big|_{\partial\Omega} = 0$, problem (4.3) has a unique solution

$$u \in C^2\left([0,T]; L^p(\Omega)\right) \cap C^1\left([0,T]; W^{2,p}(\Omega)\right); \tag{4.11}$$

moreover,

$$\Delta u \in C([0,T]; W^{2,p}(\Omega)),$$
$$\partial_t^2 u \in C^\alpha\left([\varepsilon,T]; L^p(\Omega)\right), \quad \partial_t u,\ \Delta u \in C^\alpha([\varepsilon,T]; W^{2,p}(\Omega)), \qquad \varepsilon \in (0,T),$$

and for $t \in [0,T]$,

$$\begin{aligned} &\left\|\partial_t^2 u(t,\cdot)\right\|_{L^p(\Omega)} + \|\partial_t u(t,\cdot)\|_{W^{2,p}(\Omega)} + \|\Delta u(t,\cdot)\|_{W^{2,p}(\Omega)} \\ \le\ &\text{const}\ \left(\|g\|_{C^\alpha([0,T];W^{2,p}(\partial\Omega))} + \sum_{j=0}^{1} \|\varphi_j\|_{W^{2,p}(\Omega)} + \|\Delta\varphi_j\|_{W^{2,p}(\Omega)} \right). \end{aligned}$$

Here, for getting the uniqueness we used the fact that if u is a solution of problem (4.3) satisfying (4.11), then $(\partial_t u(t,\cdot))\big|_{\partial\Omega} = \partial_t\left(u(t,\cdot)\big|_{\partial\Omega}\right)$, by virtue of the isomorphism $\mathcal{P}$, and therefore

$$x'(t) = P_1 u'(t) \ \ (t \in [0,T]), \quad x(0) = \ \varphi_0\big|_{\partial\Omega}, \quad x'(0) = \varphi_1\big|_{\partial\Omega}.$$

Remark 4.4. To our knowledge, the result in Example 4.3 is new even for the case of $p = 2$ and $a = 0$.

Example 4.5. Let $\rho > 0$, $\alpha \in (0,1)$, $f \in C^\alpha([0,T]; C[0,1])$,

$$g_j,\ h_j \in C^\alpha([0,T]; \mathbf{C}), \quad j = 0, 1.$$

For each $i,\ j = 0,\ 1$, let $\mathcal{A}_{ij}(\partial_\xi)$ (resp. $\mathcal{B}_{ij}(\partial_\xi)$) be a linear differential operator in $[0,1]$ with complex coefficients of the order not exceeding 3 (resp. of the order 1). We consider a damped Euler-Bernoulli beam equation with dynamic boundary

conditions:
(4.12)

$$\begin{cases} \partial_t^2 u + \partial_\xi^4 u - \rho \partial_\xi^2 \partial_t u = f, & \text{in } (0,T]\times[0,1], \\ \partial_t^2 u(t,j) + \mathcal{A}_{0j}(\partial_\xi)u(t,j) + \mathcal{B}_{0j}(\partial_\xi)\partial_t u(t,j) = g_j, & \text{in } (0,T]\times\{0,1\}, \\ \partial_t^2\partial_\xi^2 u(t,j) + \mathcal{A}_{1j}(\partial_\xi)u(t,j) + \mathcal{B}_{1j}(\partial_\xi)\partial_t u(t,j) = h_j, & \text{in } (0,T]\times\{0,1\}, \\ u(0,\cdot) = \varphi_0, \quad \partial_t u(0,\cdot) = \varphi_1, & \text{in } [0,1], \\ \partial_t \partial_\xi^2 u(0,j) = \psi_j, & j = 0,1. \end{cases}$$

Take

$$E = C[0,1], \quad X = \mathbf{C}^4,$$

$$A = \frac{d^4}{d\xi^4} \quad \text{with} \quad \mathcal{D}(A) = C^4[0,1],$$

$$B = -\rho \frac{d^2}{d\xi^2} \quad \text{with} \quad \mathcal{D}(B) = C^2[0,1],$$

$$G_0\varphi = -\begin{pmatrix} \mathcal{A}_{00}(\partial_\xi)\varphi(0) \\ \mathcal{A}_{01}(\partial_\xi)\varphi(1) \\ \mathcal{A}_{10}(\partial_\xi)\varphi(0) \\ \mathcal{A}_{11}(\partial_\xi)\varphi(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(G_0) := \mathcal{D}(A),$$

$$G_1\varphi = -\begin{pmatrix} \mathcal{B}_{00}(\partial_\xi)\varphi(0) \\ \mathcal{B}_{01}(\partial_\xi)\varphi(1) \\ \mathcal{B}_{10}(\partial_\xi)\varphi(0) \\ \mathcal{B}_{11}(\partial_\xi)\varphi(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(G_1) := \mathcal{D}(B),$$

$$P\varphi = \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi''(0) \\ \varphi''(1) \end{pmatrix} \quad \text{for} \quad \varphi \in \mathcal{D}(P) := \mathcal{D}(A),$$

$$P_1\varphi = \left\{ \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ z_3 \\ z_4 \end{pmatrix}; \quad z_3, \ z_4 \in \mathbf{C} \right\} \quad \text{for} \quad \varphi \in \mathcal{D}(P_1) := \mathcal{D}(B),$$

$$A_1 = 0, \quad B_1 = 0, \quad \widetilde{A} = 0, \quad \widetilde{A}_1 = 0, \quad \widetilde{B} = 0, \quad \widetilde{B}_1 = 0.$$

Then we have

$$A_0 = \frac{d^4}{d\xi^4} \ \text{ with } \ \mathcal{D}(A_0) = \{\varphi \in C^4[0,1]; \ \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0\},$$

$$B_0 = -\rho \frac{d^2}{d\xi^2} \ \text{ with } \ \mathcal{D}(B_0) = \{\varphi \in C^2[0,1]; \ \varphi(0) = \varphi(1) = 0\},$$

$$[\mathcal{D}(A)]_P \simeq C^4[0,1], \quad [\mathcal{D}(B)]_{P_1} \simeq C^2[0,1].$$

Obviously (H_1) and (H_3) are satisfied. So is (H_2) from [14, p. 1017, line 4]. Furthermore, we know that G_0 (resp. G_1) is A_0-bounded (resp. B_0-bounded) with A_0-bound (resp. B_0-bound) zero (cf. [7, p. 170]). Thus the hypotheses of Theorem 2.3 are all satisfied. Therefore Theorem 3.3 is applicable. In this case,

$$y(t) = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad u(t) = u(t,\cdot), \quad x(t) := \begin{pmatrix} u(t,0) \\ u(t,1) \\ \partial_\xi^2 u(t,0) \\ \partial_\xi^2 u(t,1) \end{pmatrix}, \quad t \in (0,T],$$

$$\overline{\mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{B})} = \left\{ \begin{pmatrix} \varphi \\ x \end{pmatrix} \in C[0,1] \times \mathbf{C}^4; \quad x \in P_1\varphi \right\}, \tag{4.13}$$

$$\boldsymbol{\Upsilon} \supset \left\{ \begin{pmatrix} \varphi \\ x \end{pmatrix} \in C^2[0,1] \times \mathbf{C}^4; \quad P\varphi = x \right\}. \tag{4.14}$$

It is not hard to verify (4.13). For (4.14), we exploit the fact (shown in the proof of [14, Theorem 5.1]) that

$$\lim_{t\to 0^+} \inf \left\{ t\|\psi\|_{C^4[0,1]} + \|\varphi-\psi\|_{C^2[0,1]} + t^{-1}\|\varphi-\psi\|_{C[0,1]}; \quad \psi \in \mathcal{D}(A_0) \right\} = 0 \tag{4.15}$$

for every

$$\varphi \in \Omega := \left\{ \psi \in C^2[0,1]; \quad \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0 \right\}.$$

Given $\varphi \in C^2[0,1]$, we put

$$\varphi_*(\xi) = \varphi(0) + (\varphi(1)-\varphi(0))\xi + \frac{1}{2}\varphi''(0)\xi^2 + \frac{1}{6}(\varphi''(1)-\varphi''(0))\xi^3, \quad \xi \in [0,1].$$

Then $\varphi - \varphi_* \in \Omega$. This in combination with (4.15) yields that

$$\lim_{t\to 0^+} \Psi\left(t, \begin{pmatrix} \varphi \\ P\varphi \end{pmatrix}\right) = 0$$

and so $\begin{pmatrix} \varphi \\ P\varphi \end{pmatrix} \in \boldsymbol{\Upsilon}$. We now use Theorem 3.3 to conclude:

(i) For every $\varphi_0 \in C^2[0,1]$, $\varphi_1 \in C^1[0,1]$, $\psi_j \in \mathbf{C}$ ($j = 0,1$), problem (4.12) has a unique solution

$$u \in \bigcap_{i=0}^{2} C^i\left((0,T]; C^{4-2i}[0,1]\right) \bigcap \left(\bigcap_{k=0}^{1} C^k\left([0,T]; C^{2-2k}[0,1]\right) \right). \tag{4.16}$$

(ii) $\partial_t^i u \in C^\alpha\left([\varepsilon,T]; C^{4-2i}[0,1]\right)$ ($\varepsilon \in (0,T)$, $i = 0, 1, 2$) and

$$\|u(t,\cdot)\|_{C[0,1]} \le \text{const} \left[\|f\|_{C([0,T];C[0,1])} + \sum_{j=0}^{1} \Big(\|g_j\|_{C([0,T];\mathbf{C})} + \|h_j\|_{C([0,T];\mathbf{C})} + |\psi_j| \Big) + \|\varphi_0\|_{C^2[0,1]} + \|\varphi_1\|_{C[0,1]} \right], \quad t \in [0,T].$$

(iii) If $\varphi_0 \in C^4[0,1]$, $\varphi_1 \in C^2[0,1]$, $\psi_j = \varphi_1''(j)$ ($j = 0,1$), and

$$\varphi_0^{(4)}(j) + \mathcal{A}_{0j}(\partial_\xi)\varphi_0(j) - \rho\varphi_1''(j) + \mathcal{B}_{0j}(\partial_\xi)\varphi_1(j) = f(0,j) - g_j, \quad j = 0,1,$$

then the solution u belongs to $\bigcap_{i=0}^{2} C^i\left([0,T]; C^{4-2i}[0,1]\right)$.

Here, for getting the uniqueness, the following fact was taken into account: if u is a solution of problem (4.12) satisfying (4.16), then

$$x(t) = Pu(t), \quad x'(t) = Pu'(t), \quad t \in (0, T],$$

$$x(0) = \begin{pmatrix} \varphi_0(0) \\ \varphi_0(1) \\ \varphi_0''(0) \\ \varphi_0''(1) \end{pmatrix}, \quad x'(0) = \begin{pmatrix} \varphi_1(0) \\ \varphi_1(1) \\ \psi_0 \\ \psi_1 \end{pmatrix}.$$

Remark 4.6. In the case of zero boundary value, i.e., when $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$, g_i, h_j, $\varphi_i(j)$, and ψ_j $(i, j = 0, 1)$ are all zero, conclusion (i) and a weaker form of conclusion (iii) are due to [14, Theorem 5.1].

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